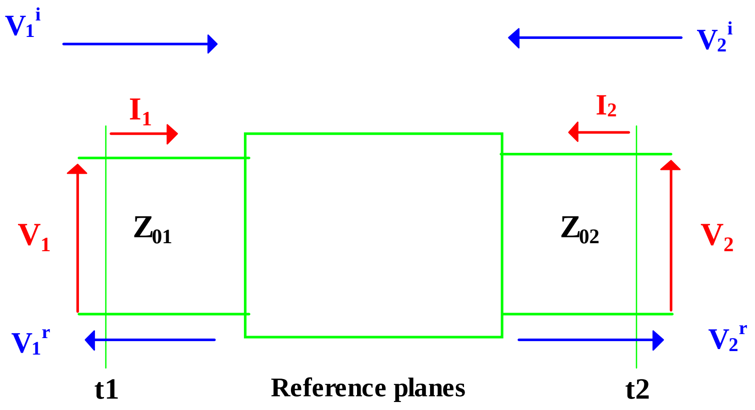
S-PARAMETERS

Scattering Matrix



Input T.L L1=0
$V1 = V_1^i + V_1^r$

Output T.L L2=0
$V2 = V_2^i + V_2^r$

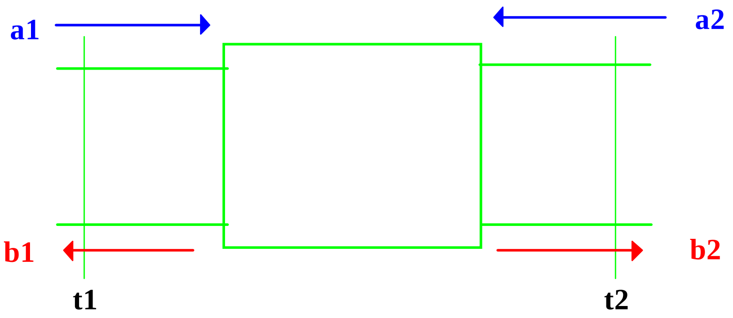
Define normalised voltages $a1 = \dfrac{V_1^i}{\sqrt{Z_{01}}}$

$$b1 = \frac{V_1^r}{\sqrt{Z_{01}}}$$

$$a2 = \frac{V_2^i}{\sqrt{Z_{01}}}$$

$$b2 = \frac{V_2^r}{\sqrt{Z_{01}}}$$

Note Using normalised voltages
≡ making the characteristic impedances
= 1 Ω



Outgoing voltage waves ∝ ingoing voltage waves .

$$b1 = S_{11}.a1 + S_{12}.a2$$
$$b2 = S_{21}.a1 + S_{22}.a2$$

$$\begin{bmatrix} b1 \\ b2 \end{bmatrix} = \begin{bmatrix} S11 & S12 \\ S21 & S22 \end{bmatrix} \begin{bmatrix} a1 \\ a2 \end{bmatrix}$$

$\uparrow$

Scattering
Matrix

Measurement of S11 & S12

(1) Apply input to Port 1 , terminate at Port 2 with a Matched Load $\therefore$a2 = 0.

$$\therefore\ S_{11} = \left.\frac{b1}{a1}\right)_{a2=0} = \text{reflection coefficient at Port 1 (matched load at Port 2).}$$

$$S_{21} = \left.\frac{b2}{a1}\right)_{a2=0} = \text{Voltage transfer ratio Port 1 to Port 2 (matched load at Port 2).}$$

(2) Apply input to Port 2 , terminate at Port 1 with a matched load $\therefore$a1 = 0.

$$\therefore\ S_{22} = \left.\frac{b2}{a2}\right)_{a1=0} = \text{reflection coefficient ar Port 2 (matched load at Port 1).}$$

$$S_{12} = \left.\frac{b1}{a2}\right)_{a1=0} = \text{transfer ratio Port 2 to Port 1 (matched load at Port 1).}$$

Note each S matrix element is in general complex eg

$$S_{i.j} = \left|S_{i.j}\right| e^{j.\theta i.j}$$

ratio of amplitudes

phase difference of signals

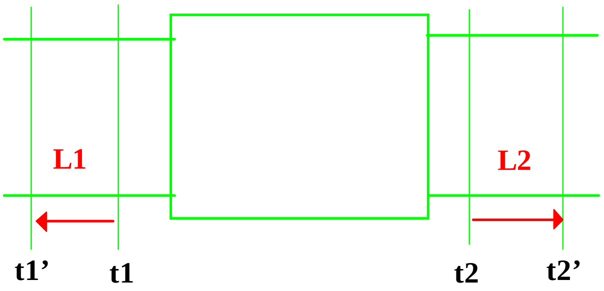
$$\text{eg } S_{11} = 0.9\angle -37 \equiv 0.9e^{j(-37)}$$

$$S_{21} = 4.0\angle 127$$

Note S11,S22 are reflection coefficients $\therefore$can plot these on a Smith Chart.

S12,S21 are not reflection coefficients and are usually plotted on a Polar diagram.

Change of reference plane



Note positive signs for L1 & L2 are taken to be movements **away** from the device

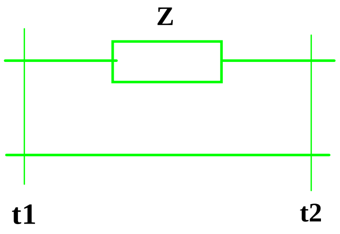
Planes t1' and t2'

$\underset{\substack{\uparrow\\ t1',t2'}}{S11'} = \underset{\substack{\uparrow\\ t1,t2}}{S11}e^{-j\,2.\beta\,L1}$ $\quad$ $S22' = S22e^{-j.2.\beta.L2}$

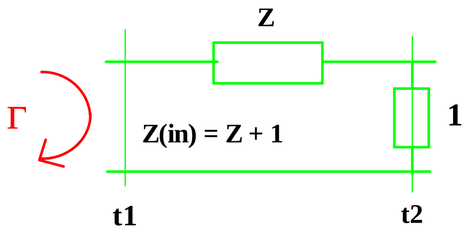
$S21' = S21e^{-j.\beta.(L1\ +L2)}$ $\quad$ $S12 = S12e^{-j.\beta.(L1\ +L2)}$

S Matrix for simple elements

Series normalised impedance



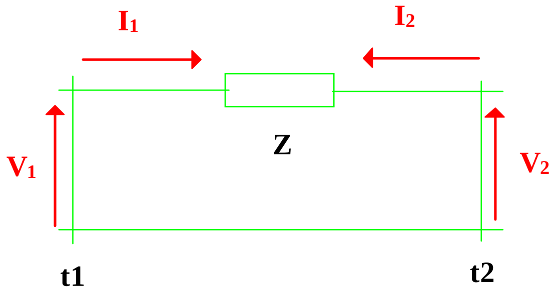
(1) Connect a matched load at t2 ie



$$\Gamma = \frac{Z_{(in)} - Z_0}{Z_{(in)} + Z_0} = \frac{Z_{(in)} - 1}{Z_{(in)} + 1} = \frac{z\ +\ 1\ -1}{z\ +1\ +1} = \frac{z}{z+2} = S_{11}$$

$$\boxed{\begin{array}{l}\text{divide by } Z_o \\ \text{to normalise}\end{array}}$$

$$S_{11} = S_{22}\ \text{(By symmetry)} = \underline{\frac{z}{z+2}}$$



$$\left.\begin{array}{ll} V1 = a1 + b1 & V2 = a2 + b2 \\ I1 = a1 - b1 & I2 = a2 - b2 \end{array}\right\} \text{General}$$

Series connection ∴ I2 = -I1.

∴ a2 - b2 = -(a1 - b1)

Matched load @ Port 2 a2 = 0

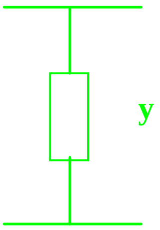
∴ b2 = a1 - b1

$$\therefore \left.\frac{b2}{a1}\right|_{a2=0} = 1 - \frac{b1}{a1} = 1 - S11 \equiv S21$$

$$S21 = 1 - S11 = 1 - \frac{z}{z+2} = \frac{2}{z+2} = S12 \text{ (by symmetry)}$$

$$\frac{z+2}{z+2} - \frac{z}{z+2} = \frac{\cancel{z -} z + 2}{z+2}$$

Shunt normalised Admittance



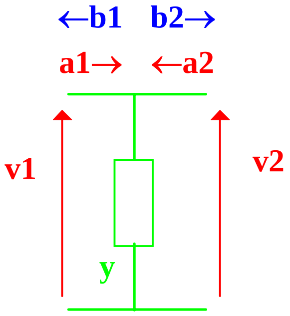
$$\text{Let } z = \frac{1}{y}$$

$$\therefore \ zin = \frac{\frac{1}{y}.1}{\frac{1}{y}+1} = \frac{\frac{1}{y}}{\frac{1}{y}+1} \quad \text{x top \& bottom by y}$$

$$\frac{1}{y+1} = Zin$$

$$\Gamma = \frac{Zin - Zo}{Zin + Zo} = \frac{\frac{1}{y+1}-1}{\frac{1}{y+1}+1} \quad \text{x top \& bottom by } y+1$$

$$\frac{1 - y + 1}{1 + y + 1} = \frac{-y}{2+y}$$



$$V1 = V2$$

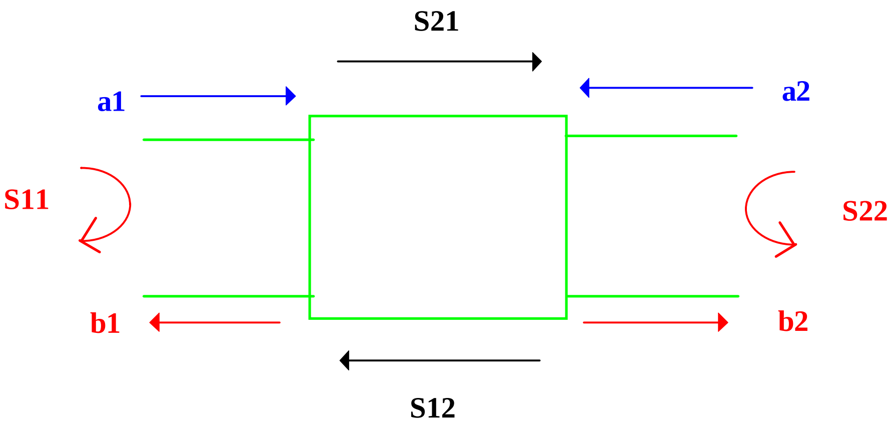
$$\therefore \ a2 + b2 = a1 + b1$$

matched load on Port 2 $\therefore \ a2 = 0$

$$S21 = \frac{b2}{a1} = \frac{a1}{a1} + \frac{b1}{a1} = 1 + S11$$

$$1 + \frac{-y}{2+y} = \frac{2+y}{2+y} + \frac{-y}{2+y} = \frac{2}{2+y} = S21$$

S Parameters for a reciprocal,lossless network



At Port 1 Power in $| a_1 |^2$

Power out $| b_1 |^2$

∴ Net power in $| a_1 |^2 - | b_1 |^2$

Reciprocal network

S21 = S12 (generally Sij = Sji)

For a lossless network Power out = Power in

$$\sum_{\text{Port}\,i} |b_1|^2 = \sum_{\text{Port}\,i} |a_1|^2$$

$$\Rightarrow [S] \text{ is a unitary matrix}$$

$$\sum_{\text{Port}\,i} S_{i.s}.S_{i.r}^{\ *} = \partial.si \ = 1 \quad \text{is} \quad s = r$$

$$= 0 \quad \text{if} \quad s \neq r$$

$$2\,\text{port} \quad i = 1,2 \quad s,r = 1\,\&\,2$$

$$S = 1, r = 1 \quad S_{11}.S_{11}^{*} + S_{21}.S_{21}^{*} = 1$$

$$|S_{11}|^2 + |S_{21}|^2 = 1$$

$$\therefore |S_{21}| = \sqrt{1 - |S_{11}|^2}$$

$$S = 2, r = 2 \quad S_{12}.S_{12}^{*} + S_{22}.S_{22}^{*} = 1$$

$$|S_{12}|^2 + |S_{22}|^2 = 1$$

$$\therefore |S_{12}| = \sqrt{1 - |S_{22}|^2}$$

If the junction is reciprocal :- $|S_{12}| = |S_{21}| \quad \therefore |S_{22}| = |S_{11}|$

$$\mathrm{S}=1;\mathrm{r}=2 \quad \mathrm{S}_{11}.S_{12}^{*}+S_{21}.S_{22}^{*}=0$$

$$\left|\mathrm{S}_{11}\right|e^{j.\vartheta_{11}}\ x\ \left|\mathrm{S}_{12}\right|e^{-j.\vartheta_{12}}+\left|\mathrm{S}_{21}\right|e^{j.\vartheta_{21}}\ x\ \left|\mathrm{S}_{22}\right|e^{-j.\vartheta_{22}}=0$$

$$\Rightarrow e^{j.(\vartheta_{12}+\vartheta_{21})}=-e^{j.(\vartheta_{11}+\vartheta_{22})}$$

$$\vartheta_{12=}\,\vartheta_{21}\quad (\,\text{reciprocal}\,)$$

$$\mathrm{e}^{\mathrm{j}(2.\vartheta_{12})}=-e^{j(\vartheta_{11}+\vartheta_{22})}$$

$$\cos 2.\vartheta_{12}=-\cos(\vartheta_{11}+\vartheta_{22})$$

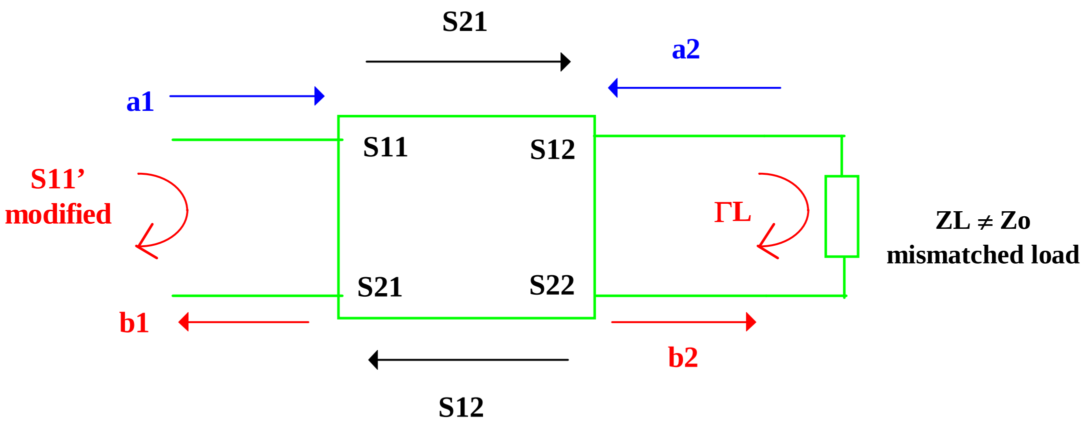
$$\sin 2.\vartheta_{12}=-\sin(\vartheta_{11}+\vartheta_{22})$$

$$\Rightarrow \underline{\vartheta_{12}=\vartheta_{21}=\frac{1}{2}(\vartheta_{11}+\vartheta_{22})+\frac{\pi}{2}\pm m.\pi \qquad \mathrm{m}=0,1,2...}$$

$\therefore$ For a reciprocal lossless 2 - port network we only need

$$\left|\mathrm{S}_{11}\right|\ ,\ \vartheta_{11}\ \&\ \vartheta_{22}$$

Modified S Parameters - mismatched source & load



$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} \qquad \Gamma_L = \frac{a_2}{b_2} \quad \therefore \quad a2 = b2.\Gamma_L$$

$$b_1 = S_{11}.a_1 + S_{12}.a_2 = S_{11}.a_1 + S_{12}.b_2.\Gamma_L$$

$$b_2 = S_{21}.a_1 + S_{22}.a_2 = S_{21}.a_1 + S_{22}.b_2.\Gamma_L$$

$$S_{11}' = \frac{b_1}{a_1} \; ; \; S_{21}' = \frac{b_2}{a_1}$$

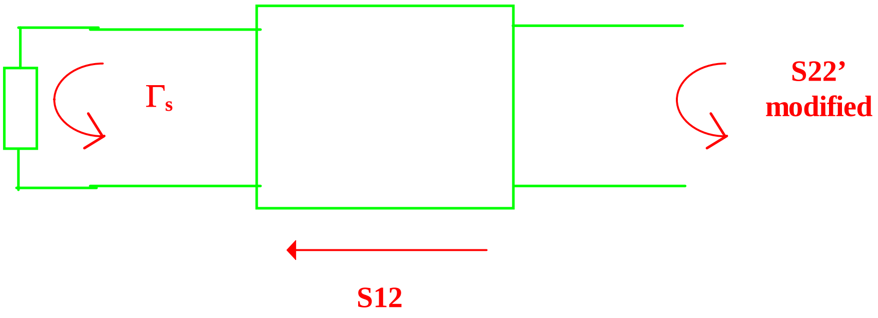
$$S_{21}' = \frac{b_2}{a_1} = \frac{S_{21}.a_1}{a_1} + \frac{S_{22}.b_2.\Gamma_L}{a1} \quad \text{rearrange gives} \quad \frac{b_2}{a_1} - \frac{S_{22}.b_2.\Gamma_L}{a1} = \frac{S_{21}.a_1}{a_1}$$

$$S_{21}'(1 - S_{22}.\Gamma_L) = S_{21} \qquad \therefore S_{21}' = \underline{\frac{S_{21}}{1 - S_{22}.\Gamma_L}}$$

$$b_1 = S_{11}.a_1 + S_{12}.\Gamma_L.\left(\frac{S_{21}}{1 - S_{22}.\Gamma_L}\right).a_1 \qquad \text{(divide by } a_1 \text{ on both sides)}$$

$$\underline{S_{11}' = S_{11} + \frac{S_{21}.S_{12}.\Gamma_L}{1 - S_{22}.\Gamma_L}}$$

Mismatched Source

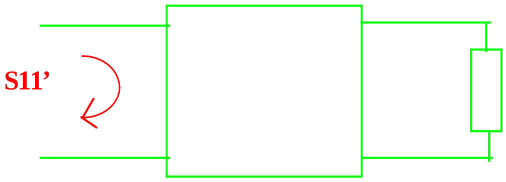


$$S_{22}' = S_{22} + \frac{S_{12} . S_{21} . \Gamma_S}{1 - S_{11} . \Gamma_S}$$

$$S_{12}' = \frac{S_{12}}{1 - S_{11} . \Gamma_S}$$

Circuit performance from S Parameters

Reflection characteristics:-



$$\text{Input V.S.W.R} = \frac{1+|S_{11}'|}{1-|S_{11}'|}$$

modulus

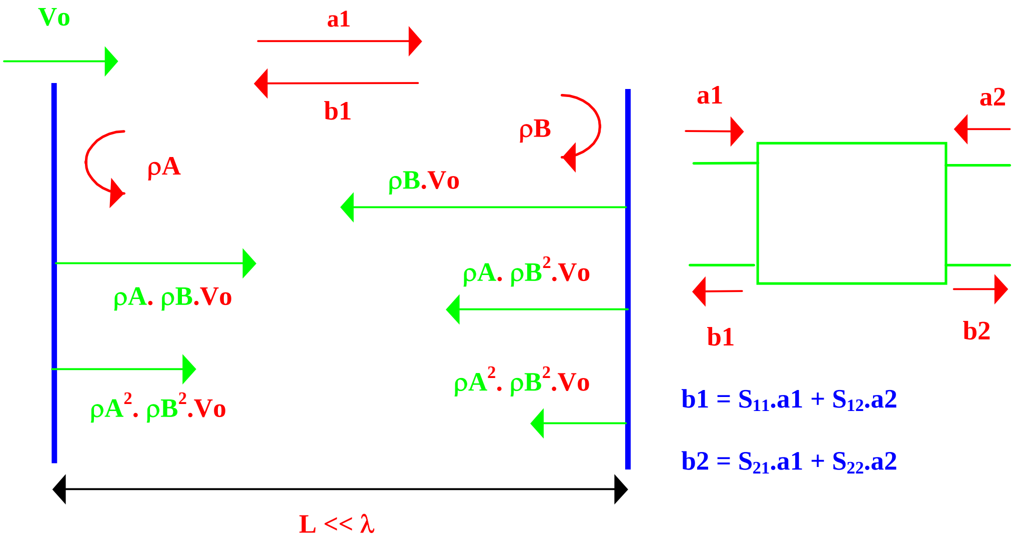
$$\text{Return Loss (dB)} = 10.\text{Log}_{10}\frac{\text{Power in}}{\text{Power reflected}}$$

$$= 10.\text{Log}_{10}\frac{|a_1|^2}{|b_1|^2} = -20.\log_{10}|S_{11}|$$

$$\frac{Z_{(in)}}{Z_o} = \frac{1+\Gamma}{1-\Gamma} = \frac{1+S_{11}'}{1-S_{11}'}$$

complex

Multiple Reflection Effects



$b1 = S_{11}.a1 + S_{12}.a2$

$b2 = S_{21}.a1 + S_{22}.a2$

$$b1 \;=\; \rho B.VB \;+\; \rho A.\rho B^2 + \rho A^2.\rho B^2.Vo + \ldots\ldots$$

$$= \rho B.Vo\left[1 + \rho A.\rho B + (\rho A.\rho B)^2 \ldots\ldots.\right]$$

$$b1 \;=\; \frac{\rho B.Vo}{1 - \rho A.\rho B} \qquad\qquad \rho B \;=\; \frac{b1}{a1} \qquad \therefore\; a1 \;=\; \frac{b1}{\rho B}$$

$$\therefore\; a1 \;=\; \frac{Vo}{(1 - \rho A.\rho B)}$$